Basic Concepts in Numerical Analysis

Larry Caretto
Mechanical Engineering 501AB

Seminar in Engineering Analysis

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California State University Northridge

Outline

- Review last class
- Midterm Exam November 15 covers material on differential equations and Laplace transforms (no phase plots)
- Overview of numerical solutions
  - Initial value problems in first-order equations
  - Systems of first order equations and initial value problems in higher order equations
  - Boundary value problems
  - Stiff systems and eigenvalues

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Review Last Class

- Phase plots, critical points, and stability
- Look at system of two linear homogenous, autonomous equations
  - d**y**/dt = **Ay** (no function of time)
- Critical points and stability depend on matrix eigenvalues which depend on determinant properties
- Described various critical points: node, center, saddle point and spiral

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Numerical Analysis Problems

- Numerical solution of algebraic equations and eigenvalue problems
- Solution of one or more nonlinear algebraic equations **f**(**x**) = **0**
- Linear and nonlinear optimization
- Constructing interpolating polynomials
- Numerical quadrature
- Numerical differentiation
- Numerical differential equations

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Interpolation

- Start with N data pairs x_i, y_i
- Find a function (polynomial) that can be used for interpolation
- Basic rule: the interpolation polynomial must fit all points exactly
- Denote the polynomial as p(x)
- The basic rule is that $p(x_i) = y_i$
- Many different forms

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Newton Polynomials

- $p(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + a_3(x - x_0)(x - x_1)(x - x_2) + \dots + a_{n-1}(x - x_0)(x - x_1)(x - x_2) \dots (x - x_{n-2})$
  - n – 1 data points numbered 0 to n – 2
- Terms with factors of $x - x_i$ are zero when $x = x_i$
  - Have $p(x_i) = y_i$ to find a_i, i = 0 to n – 1
- $a_0 = y_0$, $a_1 = (y_1 - y_0) / (x_1 - x_0)$
- $y_2 = a_0 + a_1(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1)$
  - Solve for a_2 using results for a_0 and a_1

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Newton Polynomials II

- $y_2 = a_0 + a_1(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1)$

$$a_2 = \frac{y_2 - a_0 - a_1(x_2 - x_0)}{(x_2 - x_0)(x_2 - x_1)} = \frac{y_2 - y_0 - \frac{y_1 - y_0}{x_1 - x_0}(x_2 - x_0)}{(x_2 - x_0)(x_2 - x_1)}$$

- Could continue in this fashion to determine coefficients from data
- Use alternative scheme – not derived here – known as divided difference table to compute a_k from same data

Divided Difference Table

- Enter data on x_i and y_i in rows of table skipping one row between entries
- Start with y_i data as zeroth divided difference
- First divided difference, $F_i = (y_{i+1} - y_i) / (x_{i+1} - x_i)$
  - Second (or later) divided difference is difference of first (or later) differences
  - a_i coefficients are initial divided differences

Divided Difference Table

x_0	y_0	$\leftarrow a_0$		
		$F_0 = \frac{y_1 - y_0}{x_1 - x_0}$	$\leftarrow a_1$	
x_1	y_1		$S_0 = \frac{F_1 - F_0}{x_2 - x_0}$	$\leftarrow a_2$
		$F_1 = \frac{y_2 - y_1}{x_2 - x_1}$		$T_0 = \frac{S_1 - S_0}{x_3 - x_0}$
x_2	y_2		$S_1 = \frac{F_2 - F_1}{x_3 - x_1}$	$\uparrow a_3$
		$F_2 = \frac{y_3 - y_2}{x_3 - x_2}$		$T_1 = \frac{S_2 - S_1}{x_4 - x_1}$
x_3	y_3		$S_2 = \frac{F_3 - F_2}{x_4 - x_2}$	

$F_3 = \frac{y_4 - y_3}{x_4 - x_3}$

Divided Difference Example

0	0	$\leftarrow a_0$		
		$F_0 = \frac{10-0}{10-0} = 1$	$\leftarrow a_1$	
10	10		$S_0 = \frac{3-1}{20-0} = .1$	$\leftarrow a_2$
		$F_1 = \frac{40-10}{20-10} = 3$		$T_0 = \frac{.15-.1}{30-0} = \frac{1}{600}$
20	40		$S_1 = \frac{6-3}{30-10} = .15$	$\uparrow a_3$
		$F_2 = \frac{100-40}{30-20} = 6$		
30	100			

Divided Difference Example II

- Divided difference table gives $a_0 = 0$, $a_1 = 1$, $a_2 = .1$, and $a_3 = 1/600$
- Polynomial $p(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + a_3(x - x_0)(x - x_1)(x - x_2)$ $= 0 + 1(x - 0) + 0.1(x - 0)(x - 10) + (1/600)(x - 0)(x - 10)(x - 20) = x + 0.1x(x - 10) + (1/600)x(x - 10)(x - 20)$
- Check $p(30) = 30 + .1(30)(20) + (1/600)(30)(20)(10) = 30 + 60 + 10 = 100$ (correct)

Start at Any Point in Data Table

x_{-1}	y_{-1}			
x_0	y_0	$\leftarrow a_0$		
		$F_0 = \frac{y_1 - y_0}{x_1 - x_0}$	$\leftarrow a_1$	
x_1	y_1		$S_0 = \frac{F_1 - F_0}{x_2 - x_0}$	$\leftarrow a_2$
		$F_1 = \frac{y_2 - y_1}{x_2 - x_1}$		$T_0 = \frac{S_1 - S_0}{x_3 - x_0}$
x_2	y_2		$S_2 = \frac{F_2 - F_1}{x_3 - x_1}$	$\uparrow a_3$

$F_2 = \frac{y_3 - y_2}{x_3 - x_2}$

Difference Example from x = 10

0	0			
10	10	← a_0		
		$F_0 = \frac{40-10}{20-10} = 3$	← a_1	
20	40		$S_0 = \frac{6-3}{30-10} = .15$ ← a_2	
		$F_1 = \frac{100-40}{30-20} = 6$		$T_0 = \frac{1/5-.15}{40-10} = \frac{1}{600}$
30	100		$S_1 = \frac{10-6}{40-20} = \frac{1}{5}$	↑ a_3
		$F_2 = \frac{200-100}{40-30} = 10$		

Final data point not shown is x = 40, y = 200

Divided Difference Calculation II

- Divided difference table gives $a_0 = 10$, $a_1 = 3$, $a_2 = .15$, and $a_3 = 1/600$
- Polynomial $p(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + a_3(x - x_0)(x - x_1)(x - x_2)$ $= 10 + 3(x - 10) + 0.15(x - 10)(x - 20) + (1/600)(x - 10)(x - 20)(x - 30) = x + 0.1x(x - 10) + (1/600)x(x - 10)(x - 20)$
- Check $p(40) = 10+3(30)+.15(30)(20) + (1/600)(40)(30)(20)=10+90+90+10=200$ *(correct)*

Divided Difference Code

```
for ( i = 0; i < n; i++ )
  D[0][i] = y[i];

for ( k = 1; k < n; k++ )
  for ( i = 0; i < n – k; i++ )
    D[k][i] = ( D[k-1][i+1] –
 D[k-1][i] ) / ( x[i+k] – x[i] );
```

- D[k][i] is i^{th} value of k^{th} divided difference
- Code for n data points (0 to n–1)

Constant Step Size

- Divided differences work for equal or unequal step size in x
- If $\Delta x = h$ is a constant we have simpler results
  - $F_k = \Delta y_k/h = (y_{k+1} - y_k)/h$
  - $S_k = \Delta^2 y_k/2h^2 = (y_{k+2} - 2y_{k-1} + y_k)/2h^2$
  - $T_k = \Delta^3 y_k/6h^3 = (y_{k+3} - 3y_{k+2} + 3y_{k+1} - y_k)/6h^3$
  - $\Delta^n y_k$ is called the n^{th} forward difference
  - Can also define backwards and central differences

Interpolation Approaches

- When we have N data points how do we interpolate among them?
  - Order N-1 polynomial not good choice
  - Use piecewise polynomials of lower order (linear or quadratic)
  - Can match first and or higher derivatives where piecewise polynomials join
  - Cubic splines are piecewise cubic polynomials that match first and second derivatives [as well as (x_k, y_k) values]

Cubic Spline Overview

- Have N cubic polynomials, $a_i + b_i x + c_i x^2 + d_i x^3$, with end point of 1 polynomial the start of next, requires N + 1 data points
  - Data points numbered 0 to N with polynomials numbered 1 to N
- Need 4N equations to get N values for polynomial coefficients: a_i, b_i, c_i, and d_i
- Each polynomial fits data points at ends: $p_k(x_{k-1}) = y_{k-1}$ and $p_k(x_k) = y_k$, k = 1, N

Cubic Spline Overview II

- Have continuity of first and second derivatives: $p_{k-1}'(x_k) = p_k'(x_k)$ and $p_{k-1}''(x_k) = p_k''(x_k)$
- Matching data points gives 2N equations and derivative continuity gives 2N – 2
- Have 4N – 2 equations for 4N unknown polynomial coefficients
- Different models of end point behavior used to provide additional 2 equations

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Splines in MATLAB

- Use spline function in MATLAB to get one or more interpolated points
  - xIn is array of y data for spline fit
  - yIn is array of x data for spline fit
  - Apply spline to x which can be a single data point or an array using command below

```
>> spline(xIn, yIn, x)
```

  - Generally uses not-a-knot end slopes
- Also has routine `unmkpp` to get details of resulting spline coefficients

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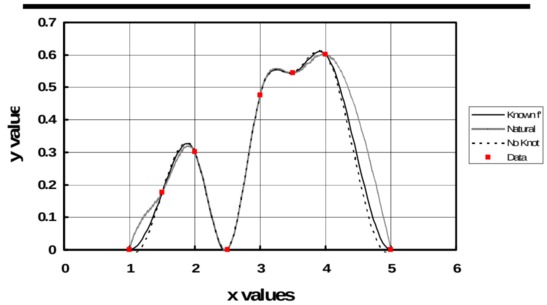
Cubic Spline Interpolation



Results show effect of different methods used to treat end points

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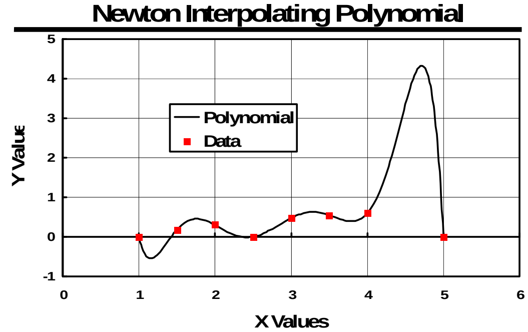
Newton Interpolating Polynomial



High-order polynomials can give unrealistic fits to data

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Polynomial Applications

- Data interpolation
- Approximation functions in numerical quadrature and solution of ODEs
- Basis functions for finite element methods
- Can obtain equations for numerical differentiation
- Statistical curve fitting (not discussed here) usually used in practice

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Derivative Expressions

- Obtain from differentiating interpolation polynomials or from Taylor series
- Series expansion for f(x) about x = a

$$f(x) = f(a) + \left.\frac{df}{dx}\right|_{x=a}(x-a) + \frac{1}{2!}\left.\frac{d^2 f}{dx^2}\right|_{x=a}(x-a)^2 + \frac{1}{3!}\left.\frac{d^3 f}{dx^3}\right|_{x=a}(x-a)^3 + \ldots$$

- Note: $d^0f/dx^0 = f$ and $0! = 1$

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!}\left.\frac{d^n f}{dx^n}\right|_{x=a}(x-a)^n$$

- What is error from truncating series?

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Truncation Error

- If we truncate series after m terms

$$f(x) = \underbrace{\sum_{n=0}^{m} \frac{1}{n!} \left.\frac{d^n f}{dx^n}\right|_{x=a} (x-a)^n}_{\text{Terms used}} + \underbrace{\sum_{n=m+1}^{\infty} \frac{1}{n!} \left.\frac{d^n f}{dx^n}\right|_{x=a} (x-a)^n}_{\text{Truncation error, } \varepsilon_m}$$

- Use theorem of mean to write truncation error as single term at unknown location, ξ, between x and a

$$\varepsilon_m = \sum_{n=m+1}^{\infty} \frac{1}{n!} \left.\frac{d^n f}{dx^n}\right|_{x=a} (x-a)^n = \frac{1}{(m+1)!} \left.\frac{d^{m+1} f}{dx^{m+1}}\right|_{x=\xi} (x-a)^{m+1}$$

Derivative Expressions

- Look at finite-difference grid with equal spacing: $h = \Delta x$ so $x_i = x_0 + ih$
- Taylor series about $x = x_i$ gives $f(x_i + kh) = f[x_0 + (i+k)h] = f_{i+k}$ in terms of $f(x_i) = f_i$

$$f(x_i + kh) = f(x_i) + \left.\frac{df}{dx}\right|_{x=x_i} kh + \frac{1}{2!}\left.\frac{d^2 f}{dx^2}\right|_{x=x_i} (kh)^2 + \frac{1}{3!}\left.\frac{d^3 f}{dx^3}\right|_{x=x_i} (kh)^3 + \ldots..$$

- Compact derivative notation

$$f_i' = \left.\frac{df}{dx}\right|_{x=x_i} \qquad f_i'' = \left.\frac{d^2 f}{dx^2}\right|_{x=x_i} \quad \cdots \quad f_{i+k}^n = \left.\frac{d^n f}{dx^n}\right|_{x=x_{i+k}}$$

Derivative Expressions II

- Combine all definitions for compact series notation

$$f(x_i + kh) = f(x_i) + \left.\frac{df}{dx}\right|_{x=x_i} kh + \frac{1}{2!}\left.\frac{d^2 f}{dx^2}\right|_{x=x_i} (kh)^2 + \frac{1}{3!}\left.\frac{d^3 f}{dx^3}\right|_{x=x_i} (kh)^3 + \ldots..$$

$$f_{i+k} = f_i + f_i' kh + \frac{f_i''(kh)^2}{2!} + \frac{f_i'''(kh)^3}{3!} + \ldots..$$

- Use this formula to get expansions for various grid locations about $x = x_i$ and use results to get derivative expressions

Derivative Expressions III

- Apply general equation for k = 1 and k = –1

$$f_{i+k} = f_i + f_i' kh + \frac{f_i''(kh)^2}{2!} + \frac{f_i'''(kh)^3}{3!} + \ldots$$

$$f_{i+1} = f_i + f_i' h + \frac{f_i'' h^2}{2!} + \frac{f_i''' h^3}{3!} + \ldots..$$

$$f_{i-1} = f_i - f_i' h + \frac{f_i'' h^2}{2!} - \frac{f_i''' h^3}{3!} + \ldots..$$

Forward

$$f_i' = \frac{f_{i+1} - f_i}{h} - \frac{f_i'' h}{2!} - \frac{f_i''' h}{3!} - \ldots.. = \frac{f_{i+1} - f_i}{h} + Ah$$

Backward

$$f_i' = \frac{f_i - f_{i-1}}{h} + \frac{f_i'' h}{2!} - \frac{f_i''' h}{3!} - \ldots.. = \frac{f_i - f_{i-1}}{h} + Ah$$

Derivative Expressions IV

- Subtract f_{i+1} and f_{i-1} expressions

$$f_{i+1} = f_i + f_i' h + \frac{f_i'' h^2}{2!} + \frac{f_i''' h^3}{3!} + \ldots..$$

$$f_{i-1} = f_i - f_i' h + \frac{f_i'' h^2}{2!} - \frac{f_i''' h^3}{3!} + \ldots..$$

$$f_{i+1} - f_{i-1} = 2 f_i' h + \frac{2 f_i''' h^3}{3!} + \frac{2 f_i''''' h^5}{5!} + \ldots..$$

$$f_i' = \frac{f_{i+1} - f_{i-1}}{2h} + \frac{f_i''' h^2}{3!} + \frac{f_i''''' h^4}{5!} - \ldots.. = \frac{f_{i+1} - f_{i-1}}{2h} + Ah^2$$

- Result called central difference expression

Order of the Error

- Forward and backward derivative have error term that is proportional to h
- Central difference error is proportional to h^2
- Error proportional to h^n called n^{th} order
- Reducing step size by a factor of α reduces n^{th} order error by α^n

$$\varepsilon_2 \approx \varepsilon_1 \left(\frac{h_2}{h_1}\right)^n$$

Order of the Error Notation

- Write the error term for n^{th} error term as $O(h^n)$
  - Big oh notation, O, denotes order
  - Recognizes that factor multiplying h^n may change slightly with h

First order forward

$$f_i' = \frac{f_{i+1} - f_i}{h} + O(h)$$

First order backward

$$f_i' = \frac{f_i - f_{i-1}}{h} + O(h)$$

Second order central

$$f_i' = \frac{f_{i+1} - f_{i-1}}{2h} + O(h^2)$$

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Higher Order Derivatives

- Add f_{i+1} and f_{i-1} expressions; solve for f_i''

$$f_{i+1} = f_i + f_i' h + \frac{f_i'' h^2}{2!} + \frac{f_i''' h^3}{3!} + ...$$

$$f_{i-1} = f_i - f_i' h + \frac{f_i'' h^2}{2!} - \frac{f_i''' h^3}{3!} + ...$$

$$f_{i+1} + f_{i-1} = 2 f_i + 2\frac{f_i'' h^2}{2!} + \frac{2 f_i'''' h^4}{4!} + \frac{2 f_i'''''' h^6}{6!} + ...$$

$$f_i'' = \frac{f_{i+1} + f_{i-1} - 2 f_i}{h^2} + \frac{f_i'''' h^2}{3!} + \frac{f_i'''''' h^4}{5!} - ..... = \frac{f_{i+1} + f_{i-1} - 2 f_i}{h^2} + O\left(h^2\right)$$

- f'' is second-order, central difference expression for second derivative

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Higher Order Directional

- We can get higher truncation error expressions at the expense of more computations
- Get second order forward and backward derivative expressions from previous results and f_{i+2} and f_{i-2}, respectively
- Combine f_{i+2} and f_{i-2} equations with previous expressions for f_{i+1} and f_{i-1} to eliminate first order error term

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Specific Taylor Series

- General equation $f_{i+k} = f_i + f_i' kh + \frac{f_i''(kh)^2}{2!} + \frac{f_i'''(kh)^3}{3!} + ...$
- k = 2 $f_{i+2} = f_i + 2 f_i' h + 4\frac{f_i'' h^2}{2!} + 8\frac{f_i''' h^3}{3!} + ...$
- k = -2 $f_{i-2} = f_i - 2 f_i' h + 4\frac{f_i'' h^2}{2!} - 8\frac{f_i''' h^3}{3!} + ...$
- k = 3 $f_{i+3} = f_i + 3 f_i' h + 9\frac{f_i'' h^2}{2!} + 27\frac{f_i''' h^3}{3!} + ...$
- k = -3 $f_{i-3} = f_i - 3 f_i' h + 9\frac{f_i'' h^2}{2!} - 27\frac{f_i''' h^3}{3!} + ...$

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Second Order Forward

- Subtract $4f_{i+1}$ from f_{i+2} to eliminate h^2 term

$$f_{i+2} - 4 f_{i+1} = \left[f_i + 2 f_i' h + 4 f_i'' \frac{h^2}{2} + 8 f_i''' \frac{h^3}{6} + ... \right]$$

$$-4\left[f_i + f_i' h + f_i'' \frac{h^2}{2} + f_i''' \frac{h^3}{6} + ... \right] = -3 f_i - 2 f_i' h + 4 f_i''' \frac{h^3}{6} + ...$$

$$f_{i+2} - 4 f_{i+1} + 3 f_i = -2h f_i' + 4 f_i''' \frac{h^3}{6} + ...$$

$$f_i' = \frac{-f_{i+2} + 4 f_{i+1} - 3 f_i}{2h} + f_i''' \frac{h^2}{3} + ...$$

Second order error

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Second Order Backwards

- Add $4f_{i-1}$ to $-f_{i-2}$ to eliminate h^2 term

$$-f_{i-2} + 4 f_{i-1} = -\left[f_i - 2 f_i' h + 4 f_i'' \frac{h^2}{2} - 8 f_i''' \frac{h^3}{6} + ... \right]$$

$$+4\left[f_i - f_i' h + f_i'' \frac{h^2}{2} - f_i''' \frac{h^3}{6} + .. \right] = 3 f_i - 2 f_i' h + 4 f_i''' \frac{h^3}{6} + ...$$

$$-f_{i-2} + 4 f_{i-1} - 3 f_i = -2h f_i' + 4 f_i''' \frac{h^3}{6} + ...$$

$$f_i' = \frac{f_{i-2} - 4 f_{i-1} + 3 f_i}{2h} + f_i''' \frac{h^2}{3} + ...$$

Second order error

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Other Derivative Expressions

- Can continue in this fashion
  - Write Taylor series for f_{i+1}, f_{i-1}, f_{i+2}, f_{i-2}, f_{i+3}, f_{i-3}, etc.
  - Create linear combinations with factors that eliminate desired terms
  - Eliminate f_i term to obtain central difference
  - Keep only terms in f_k with $k \geq i$ for forward difference expressions
  - Keep only terms in f_k with $k \leq i$ for forward difference expressions
  - Results in numerical analysis texts/online

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Other Derivative Formulas

$$f_i' = \frac{-11f_i + 18f_{i+1} - 9f_{i+2} + 2f_{i+3}}{2h} - f_i'''' \frac{h^3}{3} + \ldots$$

$$f_i' = \frac{11f_i - 18f_{i-1} + 9f_{i-2} - 2f_{i-3}}{2h} + f_i'''' \frac{h^3}{4} + \ldots$$

$$f_i' = \frac{f_{i-2} - 8f_{i-1} + 8f_{i+1} - f_{i+2}}{2h} + f_i''''' \frac{h^4}{30} + \ldots$$

$$f_i'' = \frac{-f_{i-2} + 16f_{i-1} - 30f_i + 16f_{i+1} - f_{i+2}}{12h^2} + f_i'''''' \frac{h^4}{90} + \ldots$$

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Order of Error Examples

- Table 1 in online notes shows error in first derivative for e^x around x = 1
  - Using first- and second-order forward and second-order central differences
  - Step h = 0.4, 0.2, and 0.1
  - Error ratio for doubling step size
    - 4.01 to 4.02 for central differences
    - 2.07 to 2.15 for first-order forward differences
    - 4.32 to 4.69 for second-order forward

$$n \approx \frac{\log\left(\varepsilon_2/\varepsilon_1\right)}{\log\left(h_2/h_1\right)} = \frac{\log(\varepsilon_2) - \log(\varepsilon_1)}{\log(h_2) - \log(h_1)} = \frac{d\log(\varepsilon)}{d\log(h)}$$

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Roundoff Error

- Possible in derivative expressions from subtracting close differences
- Example $f(x) = e^x$: $f'(x) \approx (e^{x+h} - e^{x-h})/(2h)$ and error at x = 1 is $(e^{1+h} - e^{1-h})/(2h) - e$

$$E = \frac{3.004166 - 2.722815}{2(0.1)} - 2.718282 = 4.5x10^{-3}$$

Second order error

$$E = \frac{2.7185536702 - 2.7180100139}{2(0.0001)} - 2.718281828459 = 4.5x10^{-9}$$

$$E = \frac{2.71828210028724 - 2.71828155660388}{2(0.0000001)} - 2.718281828 = 5.9x10^{-9}$$

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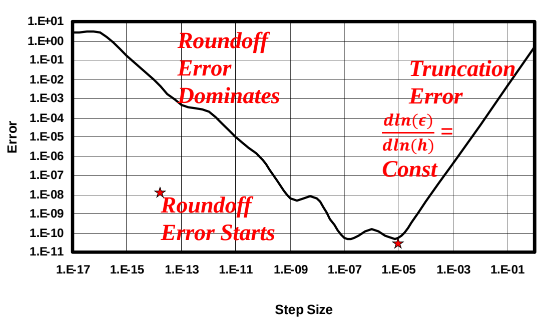
Figure 2-1. Effect of Step Size on Error



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Richardson Extrapolation

- Uses finite-difference method with two step sizes to get improved accuracy
- Start with E = F(h) + TE = F(h) + $O(h^n)$
  - E is exact result
  - F(h) is finite difference approximation with step size h
  - Truncation error, TE, is $O(h^n)$
  - Actually have an infinite series for error

$$TE = \frac{h^n}{n!}\left(\frac{d^n f}{dx^n}\right)_{x=\xi} = \sum_{k=n}^{\infty} \frac{h^n}{n!}\left(\frac{d^n f}{dx^n}\right)_{x=a} = \sum_{k=n}^{\infty} A_k h^k$$

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Richardson Extrapolation II

- Look at evaluating error with two step sizes, h and kh
  - Exact value will not change
  - Create sum to display first error term

$$E = F(h) + TE = F(h) + \sum_{m=n}^{\infty} A_m h^m = F(h) + \boxed{A_n h^n} + \sum_{m=n+1}^{\infty} A_m h^m$$

$$E = F(kh) + TE = \cdots\cdots = F(kh) + \boxed{A_n (kh)^n} + \sum_{m=n+1}^{\infty} A_m (kh)^m$$

  - Multiply first equation by k^n and subtract the second equation to eliminate the A_n term

$$k^n E - E = k^n F(h) - F(kh) + \cancel{k^n A_n h^n - A_n (kh)^n} + O(h^{n+1})$$

May be h^{n+2}

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Richardson Extrapolation III

- Solve equation from previous slide for E

$$k^n E - E = k^n F(h) - F(kh) + O(h^{n+1})$$

$$E = \frac{k^n F(h) - F(kh)}{k^n - 1} + O(h^{n+1}) = RE + O(h^{n+1})$$

- The formula for the Richardson extrapolation, RE, has a higher order of the error
  - Truncation error for RE shown below

$$TE = \frac{k^n \sum_{k=n+1}^{\infty} A_m h^m - \sum_{k=n+1}^{\infty} A_m (kh)^m}{k^n - 1} = \sum_{k=n+1}^{\infty} B_m h^m \qquad B_m = \frac{k^n - k^m}{k^n - 1} A_m$$

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Richardson Extrapolation IV

- What does this mean?

$$E = \frac{k^n F(h) - F(kh)}{k^n - 1} + O(h^{n+1}) = RE + O(h^{n+1})$$

- E is the exact result, F(h) is a finite difference result with step size h
  - If we have two n^{th}-order finite difference results, with two step sizes h and kh, we can use this formula to get an improved result with an error order of n + 1 (or higher if the error term has every other power of h)

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Richardson Extrapolation V

- Richardson extrapolation for forward dcos(x)/dx at x = 1 and h = 0.1 & h = 0.2
  - What are k and n? ***$k = h_2/h_1 = 2$; n = order = 1***

$$f_i' = \frac{f_{i+1} - f_i}{h} + O(h^1)$$

$$f_i'(h=0.1) = \frac{\cos(1.1) - \cos(1)}{0.1} = -0.8670618$$

$$f_i'(h=0.2) = \frac{\cos(1.2) - \cos(1)}{0.2} = -0.8897228$$

$$RE = \frac{k^n F(h) - F(kh)}{k^n - 1} = \frac{2^1(-0.86706) - -0.88972}{2^1 - 1} = -0.84440093$$

- Extrapolation closer to correct value of $dcos(x)/dx|_{x=1} = -\sin(1) = -0.84147098$

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Richardson Extrapolation VI

- Richardson extrapolation for central dcos(x)/dx at x = 1 and h = 0.1 & h = 0.2
  - What are k and n? ***$k = h_2/h_1 = 2$; n = order = 2***

$$f_i' = \frac{f_{i+1} - f_{i-1}}{2h} + O(h^2)$$

$$f_i'(h=0.1) = \frac{\cos(1.1) - \cos(0.9)}{2(0.1)} = -0.8400692$$

$$f_i'(h=0.2) = \frac{\cos(1.2) - \cos(0.8)}{2(0.2)} = -0.8358872$$

$$RE = \frac{k^n F(h) - F(kh)}{k^n - 1} = \frac{2^1(-0.84007) - -0.83587}{2^1 - 1} = -0.841468$$

- Extrapolation closer to correct value of $dcos(x)/dx|_{x=1} = -\sin(1) = -0.84147098$

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